Absolute Value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|x| = \sqrt{x^2}$$

Pythagorean Theorem:

In a right triangle with right angle at C, $a^2 + b^2 = c^2$

Distance Formula:

Given $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula:

Midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Quadratic Formula:

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Translations of Functions:

Replacing x with $x - h$ and y with $y - k$ will translate a function h units horizontally and k units vertically.

$y = f(x)$ translated h units horizontally and k units vertically will become $y - k = f(x - h)$ or $y = f(x - h) + k$.

Linear Functions:

Slope-intercept: $y = mx + b$
Point-slope: $y - y_1 = m(x - x_1)$

Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Parallel lines have equal slopes: $m_1 = m_2$

Perpendicular lines have negative reciprocal slopes: $m_1 = \dfrac{-1}{m_2}$

A normal line is perpendicular to a tangent line.

Circular Functions:

Given (h,k) as the center of a circle and r the radius, then the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

Area: $A = \pi r^2$
Circumference: $C = 2\pi r$

Symmetry: (Even and Odd Functions)

The graph of an equation is:

Symmetric with respect to the y-axis if replacing x with $-x$ gives an equivalent equation (for example $y = x^2$).
$f(x)$ is **even** if $f(-x) = f(x)$.

Symmetric with respect to the origin if replacing x with $-x$ and y with $-y$ gives an equivalent equation (for example $y = x^3$).
$f(x)$ is **odd** if $f(-x) = -f(x)$.

Symmetric with respect to the x-axis if replacing y with $-y$ gives an equivalent equation (for example $x = y^2$).

Existence of a Limit:

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$.

Limits:

$\lim_{x \to c} \frac{f(x)}{g(x)}$ Substitute $x = c$. If $\frac{0}{0}$, then factor, multiply by conjugate, rationalize, or simplify to one fraction. If $\frac{k}{0}$, then use infinite limit properties.

See L'Hopital's Rule

Infinite Limits:

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Look for a constant divided by a small negative number "-" or small positive number "+".

Limits at Infinity:

$\lim_{x \to -\infty} \frac{k}{x^r} = 0$ and $\lim_{x \to \infty} \frac{k}{x^r} = 0$, provided that $r > 0$.

Divide by the highest power in the denominator.

$\lim_{n \to \infty} r^n = 0$ if and only if $-1 < r < 1$.

Continuity at a Point:

Let f be defined on an open interval containing c. We say that f is continuous at c if:

(1) $f(c)$ is defined

(2) $\lim_{x \to c} f(x)$ exists

(3) $\lim_{x \to c} f(x) = f(c)$

Intermediate Value Theorem:

If f is continuous on $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.

Extreme Value Theorem:

If f is continuous on $[a,b]$, then f attains both a maximum value and a minimum value there. These values occur at an endpoint, where $f'(x) = 0$ or $f'(x)$ does not exist.

Definition of a Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided that this limit exists.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ provided that this limit exists.

Differentiability Implies Continuity:

If $f'(c)$ exists, then f is continuous at c.

Differential dy:

$$f'(x) = \frac{dy}{dx} \quad \text{implies} \quad dy = f'(x) \cdot dx$$

Local Linear Approximation:

$$y - y_1 = f'(x_1)(x - x_1)$$

Mean Value Theorem for Derivatives:

If f is continuous on a closed interval $[a,b]$ and differentiable on its interior (a,b), then there is at least one number c in (a,b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad \text{or} \qquad f(b) - f(a) = f'(c) \cdot (b - a)$$

$\dfrac{f(b) - f(a)}{b - a}$ is the **average rate of change** or **mean rate of change**.

$f'(c)$ is the **instantaneous rate of change.**

Rolle's Theorem: (Optional)

If f is continuous on a closed interval $[a,b]$ and differentiable on its interior (a,b) and if $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.

Mean Value Theorem for Integrals:

If f is continuous on a closed interval $[a,b]$, there is a number c between a and b such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{\int_a^b f(x)dx}{b-a}$$

OR

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

$f(c)$ is the **average value** or **mean value** (Average y value)

Average Value of a Function:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Riemann Sums: Left, Right, Midpoint (n intervals from a to b):

$$\sum_{i=1}^{n} f(x_i)\Delta x \quad , \quad \text{where } \Delta x = \frac{b-a}{n} \quad \text{and} \quad a \le x_i \le b$$

Trapezoid Rule: Average of left hand and right hand Riemann Sum.

Definition of a Definite Integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad , \quad \text{where } \Delta x = \frac{b-a}{n} \quad \text{and} \quad a \le x_i \le b$$

Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a) \quad , \quad \text{where } F'(x) = f(x)$$

OR

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \quad \text{and} \quad \frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))\cdot g'(x)$$

Increasing and Decreasing:

$f'(x) > 0$ implies $f(x)$ is increasing.
$f'(x) < 0$ implies $f(x)$ is decreasing.

Use to determine local and/or global maximum and minimum values.

Critical Points: Where $f'(x) = 0$ or $f'(x)$ does not exist or at endpoints.

The First Derivative Test for Local Extrema:

Relative minimum when f' changes from negative to positive.
Relative maximum when f' changes from positive to negative.

Concave Up and Concave Down:

$f''(x) > 0$ implies $f(x)$ is concave up.
$f''(x) < 0$ implies $f(x)$ is concave down.

Use to determine points of inflection.

$f(x)$ is concave up if $f'(x)$ is increasing.
$f(x)$ is concave down if $f'(x)$ is decreasing.

The Second Derivative Test for Local Extrema:

If $f'(c) = 0$ for some c and $f''(c) < 0$, then $f(c)$ is a local maximum.

If $f'(c) = 0$ for some c and $f''(c) > 0$, then $f(c)$ is a local minimum.

Vertical Asymptotes:

If $f(c)$ is undefined then evaluate the $\lim_{x \to c} f(x)$ to determine function behavior near c. If the $\lim_{x \to c^-} f(x) = \pm\infty$ or $\lim_{x \to c^+} f(x) = \pm\infty$, then $x = c$ is a vertical asymptote.

Horizontal Asymptotes:

Evaluate the $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$. If either limit equals a unique number L, then $y = L$ is a horizontal asymptote.

Differentiation Rules:

Let u and v be differentiable functions of x.

Constant Multiple Rule:

$$\frac{d}{dx}(ku) = k \cdot \frac{du}{dx}$$

Sum or Difference Rule:

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Product Rule:

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Constant Rule:

$$\frac{d}{dx}(k) = 0$$

Power Rule:

$$\frac{d}{dx}(u^r) = r \cdot u^{r-1} \cdot \frac{du}{dx}$$

Chain Rule:

$$\frac{d}{dx}\left(f\left(u\right)\right) = f'\left(u\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(f(g(h(x)))\right) = f'\left(g(h(x))\right) \cdot g'(h(x)) \cdot h'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Trigonometric Rules:

$$\frac{d}{dx}\left(\sin u\right) = \left(\cos u\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\csc u\right) = \left(-\csc u \cdot \cot u\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos u\right) = \left(-\sin u\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\sec u\right) = \left(\sec u \cdot \tan u\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan u\right) = \left(\sec^2 u\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cot u\right) = \left(-\csc^2 u\right) \cdot \frac{du}{dx}$$

Integration Rules:

$\int f'(x)dx = f(x) + C$ **OR** $\int f(x)dx = F(x) + C$, where $F'(x) = f(x)$

$$\int [f(x)]^r f'(x)dx = \frac{[f(x)]^{r+1}}{r+1} + C$$

$\int u^r du = \frac{u^{r+1}}{r+1} + C$ **u-substitution** $\int e^u du = e^u + C$

$\int \sin u \ du = -\cos u + C$ $\int \frac{1}{u} du = \ln|u| + C$

$\int \cos u \ du = \sin u + C$ $\int a^u du = \frac{a^u}{\ln a} + C$

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

If f is an even function:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If f is an odd function:

$$\int_{-a}^{a} f(x)dx = 0$$

Length of a Curve: $\left(y = f(x)\right)$

$$L = \int_{a}^{b} \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_{a}^{b} \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Length of a Curve, Parametric Equations: $\left(x = f(t), y = g(t)\right)$

$$L = \int_{a}^{b} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt = \int_{a}^{b} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

Work Equals Force Times Distance: (Optional)

$$W = \int_{a}^{b} F(x)dx$$

Hooke's Law – Springs: (Optional)

$F(x) = kx$ Force is directly proportional to distance stretched or compressed.

Law of Gravitation: (Optional)

$F(x) = \dfrac{k}{x^2}$ Force is inversely proportional to the square of the distance.

Let $s(t)$ equal position at time t, $v(t)$ equal velocity, and $a(t)$ equal acceleration:

$s'(t) = v(t)$ and $\int v(t)dt = s(t) + C$

Displacement or change of position: $\int_a^b v(t)dt = s(b) - s(a)$

Starting position + displacement = Ending Position: $s(a) + \int_a^b v(t)dt = s(b)$

Total Distance: $\int_a^b |v(t)|dt$

$s''(t) = v'(t) = a(t)$ and $\int a(t)dt = v(t) + C$

Change of velocity: $\int_a^b a(t)dt = v(b) - v(a)$

Starting velocity + change in velocity = ending velocity: $v(a) + \int_a^b a(t)dt = v(b)$

Average Velocity and Instantaneous Velocity:

$$v_{ave} = \frac{s(b) - s(a)}{b - a} \quad \text{and} \quad v = s'(t)$$

Exponential Functions:

Definition:

$$y = a^x \ , \ a > 0, \ a \neq 1$$

Properties:

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$\left(a^m\right)^n = a^{mn}$$

$$\left(ab\right)^m = a^m \cdot b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Logarithmic Functions:

Definition:

$\log_a x = y$ if and only if $a^y = x \ , \ x > 0, \ a > 0, \ a \neq 1$

Properties:

$$\log_a 1 = 0$$

$$\log_a \left(mn\right) = \log_a m + \log_a n$$

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a \left(m^r\right) = r \cdot \log_a m$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Inverse Functions:

$f(x) = a^x$ and $f^{-1}(x) = \log_a x$ are inverse functions

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

$a^{\log_a x} = x$ for every $x > 0$

$\log_a\left(a^x\right) = x$ for every $x \in \mathbb{R}$

Natural Logarithm: (Optional)

Definition:

$$\ln x = \int_1^x \frac{1}{t}dt \quad \text{, where } x > 0$$

The Number e: (Optional)

e is the number such that $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$

$$e = \ln^{-1} 1 = \lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}\right) \approx 2.718$$

Exponential and Logarithmic Derivatives and Integrals:

$$\frac{d}{dx}\left(a^u\right) = a^u \cdot \ln a \cdot \frac{du}{dx} \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\frac{d}{dx}\left(e^u\right) = e^u \cdot \frac{du}{dx} \qquad \int e^u\,du = e^u + C$$

$$\frac{d}{dx}\left(\log_a u\right) = \frac{1}{u\ln a} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\ln u\right) = \frac{1}{u} \cdot \frac{du}{dx} \qquad \int \frac{1}{u}du = \ln\left|u\right| + C$$

Inverse Trigonometric Derivatives and Integrals: (Optional)

$$\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \qquad \int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\frac{d}{dx}\left(\cos^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2} \cdot \frac{du}{dx} \qquad \int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\frac{d}{dx}\left(\sec^{-1} u\right) = \frac{1}{\left|u\right|\sqrt{u^2-1}} \cdot \frac{du}{dx} \qquad \int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\left(\frac{\left|u\right|}{a}\right) + C$$

Inverse Trigonometric Functions:

$$\frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$0 \le \cos^{-1} x \le \pi$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\frac{-\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

Exponential Growth/Decay Model:

$\frac{dP}{dt} = k \cdot P$, the rate of growth is proportional to the population size P.

$$P = P_0 \cdot e^{kt}$$

Logistic Growth Model:

$\frac{dP}{dt} = k \cdot P(L - P)$ **or** $\frac{dP}{dt} = k \cdot P\left(1 - \frac{P}{L}\right)$

The rate of growth is proportional to both the population size P and to the difference $(L - P)$ or $\left(1 - \frac{P}{L}\right)$, where L is the maximum population size supported or carrying capacity.

Compound Interest Formula: (Optional)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Continuous Compounding: (Optional)

$$A = Pe^{rt}$$

Hyperbolic Trigonometric Functions: (Optional)

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx}\left(\sinh u\right) = \cosh u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cosh u\right) = \sinh u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\tanh u\right) = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

Integration by Parts:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

L'Hopital's Rule:

Suppose that f **and** g **are differentiable and** $g'(c) \neq 0$. **Suppose that**

$\lim_{x \to c} f(x) = 0$ **and** $\lim_{x \to c} g(x) = 0$ **or that**

$\lim_{x \to c} f(x) = \pm\infty$ **and** $\lim_{x \to c} g(x) = \pm\infty$ $\quad \frac{0}{0}$ **or** $\frac{\infty}{\infty}$

Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

Improper Integrals with Infinite Limits:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx$$

$$\int_{a}^{\infty} f(x)dx = \lim_{b\to\infty}\int_{a}^{b} f(x)dx$$

If the limits on the right exist and have finite values, then we say the corresponding improper integrals converge and have those values. Otherwise, the integrals are said to diverge.

If both $\int_{-\infty}^{0} f(x)dx$ **and** $\int_{0}^{\infty} f(x)dx$ **converge, then** $\int_{-\infty}^{\infty} f(x)dx$ **is said to converge and have value**

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

Otherwise, $\int_{-\infty}^{\infty} f(x)dx$ **diverges.**

Improper Integrals with Infinite Integrands:

Suppose $\lim_{x\to b^-}|f(x)| = \infty$. **Then** $\int_{a}^{b} f(x)dx = \lim_{t\to b^-}\int_{a}^{t} f(x)dx$ **provided the limit exists and is finite.**

Suppose $\lim_{x\to a^+}|f(x)| = \infty$. **Then** $\int_{a}^{b} f(x)dx = \lim_{t\to a^+}\int_{t}^{b} f(x)dx$ **provided the limit exists and is finite.**

Suppose $\lim_{x\to c}|f(x)| = \infty$ **and** $a < c < b$.

Then $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$ **provided both integrals on the right converge.**

Definition of nth Taylor Polynomial and Maclaurin Polynomial:

If f has n derivatives at a, then the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the **nth Taylor Polynomial for f at a.** If $a = 0$, then

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \dots + \frac{f^{(n)}(0)}{n!}(x)^n$$

is called the **nth Maclaurin Polynomial for f at a.**

Taylor's Formula with Remainder:

Let f be a function whose $(n+1)^{\text{th}}$ derivative, $f^{(n+1)}(x)$, exists for each x in an open interval I containing a.

Then for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder (or error) $R_n(x)$ is given by the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ **(Lagrange Error)**

and c is some point between x and a.

Sequences:

A sequence is monotonic if it is strictly increasing or decreasing.

$\lim_{n\to\infty} r^n = 0$ if and only if $-1 < r < 1$.

A sequence $\{a_n\}$ converges to L if $\lim_{n\to\infty} a_n = L$.

Series:

<u>N[th] Term Test for Divergence</u>

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Geometric Series</u>

If $-1 < r < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$

converges to $S = \dfrac{a}{1-r}$, otherwise it diverges.

<u>P-Series</u>

If $p > 1$, then $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges, otherwise it diverges. (**Harmonic** if $p = 1$)

<u>Collapsing or Telescoping Series</u>

If $\lim_{n\to\infty} b_n = L$, then $\sum_{n=1}^{\infty} \left(b_n - b_{n+1}\right) = b_1 - L$.

<u>Ratio Test</u> – Good when a_n contains $n!$, r^n, or n^n.

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and suppose $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \rho$ **(rho)**

If $\rho < 1$, then the series converges.

If $\rho > 1$, then the series diverges.

If $\rho = 1$, then the test is inconclusive.

Root Test (Optional)

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$.

If $\rho < 1$, then the series converges.

If $\rho > 1$, then the series diverges.

If $\rho = 1$, then the test is inconclusive.

Limit Comparison Test – Good when a_n contains only constant powers of n.

Suppose $a_n \geq 0$, $b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and diverge together.

* If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum_{n=1}^{\infty} b_n$ convergent implies $\sum_{n=1}^{\infty} a_n$ convergent.

* If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $\sum_{n=1}^{\infty} b_n$ divergent implies $\sum_{n=1}^{\infty} a_n$ divergent.

* **(Optional)**

Ordinary or Direct Comparison

Suppose $0 \leq a_n \leq b_n$ for $n \geq N$.

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Integral Test

Let f be continuous, positive and decreasing function on the interval $[1,\infty)$ and suppose $a_k = f(k)$ for all positive integers k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

Remainder or error for the first n terms: $0 < R_n < \int_n^{\infty} f(x)dx$

Alternating Series Test

Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + ...$ be an alternating series with $a_n > a_{n+1} > 0$. If $\lim_{n\to\infty} a_n = 0$, then the series converges. The **error** made by using the sum S_n of the first n terms to approximate the sum S of the series is not more than a_{n+1}. $|S - S_n| \le a_{n+1}$

Absolute Convergence Test

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges. A series $\sum_{n=1}^{\infty}a_n$ is said to **converge absolutely** if $\sum_{n=1}^{\infty}|a_n|$ converges. . Rearrangement of terms gives the same sum.

Absolute Ratio Test

Let $\sum_{n=1}^{\infty}a_n$ be a series of nonzero terms and $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \rho$.

If $\rho < 1$, then the series converges absolutely (hence converges).

If $\rho > 1$, then the series diverges.

If $\rho = 1$, then the test is inconclusive.

Conditional Convergence

A series $\sum_{n=1}^{\infty}a_n$ is called **conditionally convergent** if $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Rearrangement of terms can give a different sum or make the series diverge.

Power Series:

Given $f(x) = \sum_{n=0}^{\infty} a_n \left(x - a\right)^n$

The domain (**interval of convergence**) of $f(x)$ is one of the following:

1. The single point $x = a$. The **radius of convergence** is 0.
2. An interval $\boldsymbol{(a - R, a + R)}$, plus possibly one or both endpoints. The **radius of convergence** is $\boldsymbol{R}$.
3. All real numbers. The **radius of convergence** is ∞.

Use the **Absolute Ratio Test** to determine the **interval of convergence.**

If f is represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n \left(x - a\right)^n$ for all x in an open interval I containing a, then $a_n = \dfrac{f^{(n)}(a)}{n!}$.

Definition of Taylor and Maclaurin Series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\left(x - a\right)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Convergence of Taylor and Maclaurin Series: (Use Absolute Ratio Test)

$f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}\left(x - a\right)^n$ if and only if there exists a c between x and a such that $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = 0$ for every x in I.

Maclaurin Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ , \ x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \ , \ x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \ , \ x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \ , \ -1 < x < 1$$

Polar Coordinates:

$$(x, y) \rightarrow (r, \theta) \qquad r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$(r, \theta) \rightarrow (x, y) \qquad x = r\cos\theta$$

$$y = r\sin\theta$$

Polar Equations: Derivatives:

Given $r = f(\theta)$, $x = r\cos\theta$ and $y = r\sin\theta$

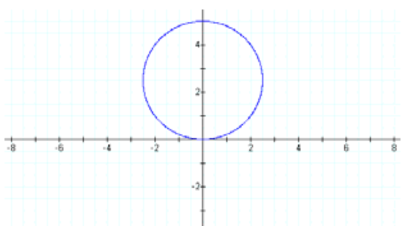
or

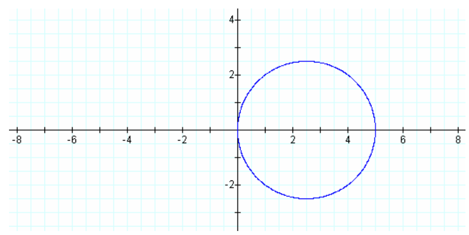
$x = f(\theta)\cdot\cos\theta$ and $y = f(\theta)\cdot\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)\cdot\cos\theta + \sin(\theta)\cdot f'(\theta)}{f(\theta)\cdot(-\sin\theta) + \cos(\theta)\cdot f'(\theta)} \quad \text{(Product Rule)}$$

Polar Equations: Area

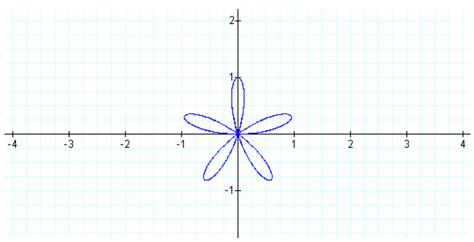
Given $r = f(\theta)$: $$A = \frac{1}{2}\int_a^b \left[f(\theta) \right]^2 d\theta$$

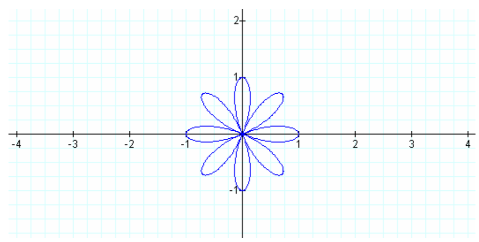




$r = 5\sin\theta$ Circles $r = 5\cos\theta$

$r = a\sin\theta \ , \ 0 \le \theta \le \pi$
$r = a\cos\theta \ , \ 0 \le \theta \le \pi$

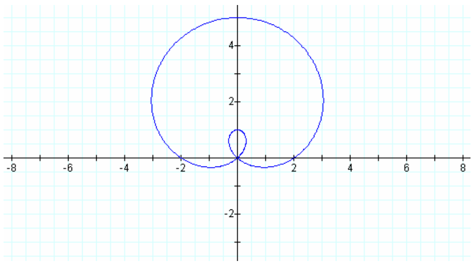


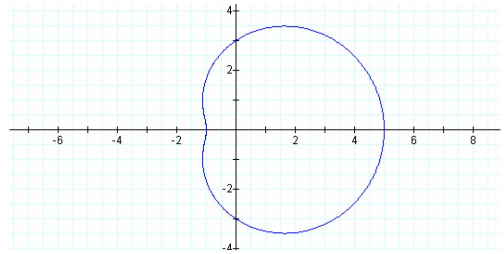


$r = \sin 5\theta$ Roses $r = \cos 4\theta$

$r = a\sin b\theta$
$r = a\cos b\theta$

If b is odd then, b leaves and $0 \le \theta \le \pi$
If b is even then, $2b$ leaves and $0 \le \theta \le 2\pi$

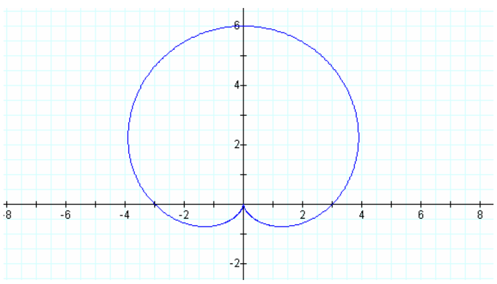




$r = 2 + 3\sin\theta$ <u>Limacons</u> $r = 3 + 2\cos\theta$

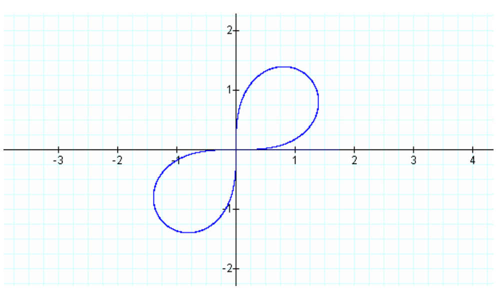
$|a| < |b|$ $|a| > |b|$

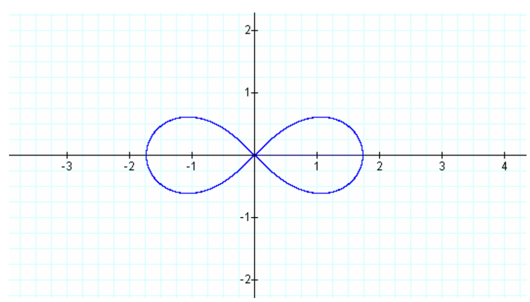
$$r = a + b\sin\theta \ , \ 0 \le \theta \le 2\pi$$
$$r = a + b\cos\theta \, , \ 0 \le \theta \le 2\pi$$



$r = 3 + 3\sin\theta$ <u>Cardioid</u>

$|a| = |b|$





$r^2 = 3\sin 2\theta$ <u>Lemniscate</u> $r^2 = 3\cos 2\theta$

$$r^2 = a\sin 2\theta$$
$$r^2 = a\cos 2\theta$$

Parametric Equations:

Given $x = f(t)$ and $y = g(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \textbf{and} \quad \frac{d^2 y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Parametric Arc Length:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Vector Valued Functions:

Given $\mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$

Velocity = $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j}$

Acceleration = $\mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t) \rangle = x''(t)\mathbf{i} + y''(t)\mathbf{j}$

Speed = $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Solving Differential Equations:

Separate Variables and Integrate (Analytically)

Slope Fields or Directional Fields (Graphical Approximations)

Euler's Method (Numerical Approximations)

Given $\frac{dy}{dx} = y' = f\left(x, y\right)$ and a point $\left(x_n, y_n\right)$, find $\left(x_{n+1}, y_{n+1}\right)$

$$x_{n+1} = x_n + \Delta x$$

$y_{n+1} = y_n + m \cdot \Delta x$ **or** $y_{n+1} = y_n + f\left(x_n, y_n\right) \cdot h$

$h = \Delta x$ is the step size